Eur. Phys. J. B **61**, 89–93 (2008)
DOI: 10.1140/epjb/e2008-00033-1

THE EUROPEAN PHYSICAL JOURNAL B

Quantum fluctuations in a scale-free network-connected Ising system

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Received 13 September 2007 / Received in final form 16 November 2007
Published online 23 January 2008 –

Abstract. We study the effect of quantum fluctuations in an Ising spin system on a scale-free network of degree exponent $\gamma > 5$ using a quantum Monte Carlo simulation technique. In our model, one can adjust the magnitude of the magnetic field *perpendicular* to the Ising spin direction and can therefore control the strength of quantum fluctuations for each spin. Our numerical analysis shows that quantum fluctuations reduce the transition temperature T_c of the ferromagnetic-paramagnetic phase transition. However, the phase transition belongs to the same mean-field type universality class both with and without the quantum fluctuations. We also study the role of hubs by turning on the quantum fluctuations exclusively at the nodes with the most links. When only a small number of hub spins fluctuate quantum mechanically, T_c decreases with increasing magnetic field until it saturates at high fields. This effect becomes stronger as the number of hub spins increases. In contrast, quantum fluctuations at the same number of "non-hub" spins do not affect T_c. This implies that the hubs play an important role in maintaining order in the whole network.

PACS. 64.60.Cn Order-disorder transformations; statistical mechanics of model systems – 64.60.Fr Equilibrium properties near critical points, critical exponents – 89.75.Hc Networks and genealogical trees

1 Introduction

In recent years, there has been a rapidly growing interest in the study of complex networks. When appropriately simplified, a great number of both natural and man-made systems can be understood in terms of a large array of objects interacting with one another in a complicated network-like manner. To name just a few examples, the Internet [1], the World-Wide Web [2,3], collaboration networks [4], citation networks [5], protein interaction [6], food-webs [7,8], and metabolic networks [9] are known to be well described using theories based on complex networks.

Of the most-studied complex networks, scale-free (SF) networks in particular are known to describe a variety of systems in the real world and to possess many peculiar and interesting properties. A node in a SF network is characterized by the degree k, the number of links attached to it. The defining characteristic of a SF network is the degree distribution $P(k)$ that has a power-law decay for a large k, i.e.,

$$P(k) \sim k^{-\gamma}. \tag{1}$$

Here γ is an important parameter of the SF network and is called the degree exponent (which is to be distinguished from the susceptibility critical exponent $\bar{\gamma}$ in this paper). The above form of the degree distribution implies that a few nodes have a very large number of links compared to the average degree of the network. When the nodes interact with one another through links, these "hubs" play an important role by influencing a large number of nodes and by enabling communication between them.

Cooperative phenomena in the SF networks have also been studied recently in many contexts, such as the spreading of epidemics [10], percolation [11], and the Ising model [12–15]. These models have all been shown to display unique characteristics due to the peculiar topology of SF networks. For example, a ferromagnetic Ising system, when placed on a SF network, is known to exhibit an extraordinary ferromagnetic-paramagnetic phase transition. The transition belongs to the simple mean-field universality class when $\gamma > 5$. However, the critical exponents for $3 < \gamma \leq 5$ are nonuniversal and depend on the value of γ, though they are independent of the details of the model used. If $\gamma \leq 3$, the system remains ordered in the ferromagnetic phase at all temperatures.

Although most of the work in complex networks has focused on classical dynamics, quantum mechanical problems in complex networks such as the localization-delocalization transitions of electronic states [16,17], the ferromagnetic-paramagnetic phase transition in an Ising model [18], and quantum gauge theory [19], have also been subjects of recent studies. Another motivation for studying quantum effects on complex networks is furnished by the recent development of broad interest in quantum computing and information processing. For example, an array

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of quantum mechanical spins in a network may form quantum bits(qubits) and the quantum information in them is usually processed and transmitted via interactions between them. Although our model in this work may not directly represent a real quantum computer, it may provide an insight into how the topology of the underlying network affects the behaviour of the qubits.

The main topic of this work — quantum fluctuations in SF networks — has drawn our attention for several reasons. Firstly, the SF nature of the networks is known to enhance correlations between Ising spins. Quantum fluctuations, on the other hand, tend to destroy correlations. In fact, a recent study on small-world networks — another widely studied class of complex networks — has shown that strong quantum fluctuations indeed destroy ferromagnetic order even at zero temperature [18]. Therefore, it would be interesting to find out how competition between the two affects the behaviour of the system. Secondly, in a SF network, the hub spins are very important because they affect many other spins and mediate information between them. The role of the hubs may be studied more closely if one can control their effectiveness. One may achieve this by applying quantum fluctuations exclusively at the hubs. Thirdly, whether quantum fluctuations change the universality class of the phase transition is an intriguing scientific question in itself. In the case of a ferromagnetic Ising system on a small-world network [18], it was found that quantum fluctuations do not alter the universality class. It may be interesting to see whether it is the case in SF networks, too. In order to focus on the effect of quantum fluctuations on the phase transitions and the universality class to which they belong, we will restrict the analysis in this paper the simple case of $\gamma > 5$, where the model belongs to the well-known mean-field universality class.

2 Model and simulation method

The model we consider is a system of N spins with Ising interactions. Unlike most Ising models in which spins form a lattice, however, each spin is placed on a node of a SF network. A pair of spins interact with each other if and only if there is a link between the nodes they are on. In order to investigate the effect of quantum fluctuations, we also apply a magnetic field in a direction perpendicular to the Ising spin direction.

The model Hamiltonian is given by

$$H = -\sum_{i<j} J_{ij}\sigma_i^z\sigma_j^z - \sum_{i=1}^{N} \Delta_i\sigma_i^x, \tag{2}$$

where σ_i^α ($\alpha = x, y, z$) are the Pauli matrices representing the spin at the ith node. The ferromagnetic coupling between the spins at nodes i and j is determined by J_{ij}. The parameter Δ_i is the magnitude of the magnetic field at node i in the x-direction. (We use normalized units such that the Bohr magneton $\mu_B = 1$.) Note that the Ising spin direction is chosen to be the z-direction and the magnetic field Δ_i is applied *perpendicular* to the Ising spin direction. This transverse field term thus causes quantum fluctuations, as it does not commute with the first Ising spin interaction term of the Hamiltonian. The strength of the quantum fluctuations at each individual spin is therefore tuneable through Δ_i.

In this work, we consider a simple ferromagnetic case where

$$J_{ij} = \begin{cases} J > 0, & \text{if there is a link between } i \text{ and } j, \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

i.e., the coupling is a nonzero constant if and only if the two nodes are linked in the network. Note that if J_{ij} is allowed to take different values from link to link, the system would become a spin glass, which we will not discuss here. For the transverse field Δ_i, we will consider only two simple cases: (a) a uniform field over the whole network and (b) a uniform field over a certain set of nodes and no field for the rest. Case (a) will be analyzed first in the next section, and then case (b) will be considered later in Section 4 when we discuss the special role played by hubs.

In order to generate random SF networks, we have used the static model [20]. We have checked that no two nodes have more than one link between them, no link connects a node to itself, and the whole network is connected as one single cluster. Once a spin has been placed on every node of the network, we have performed quantum Monte Carlo simulations [21] using the Trotter production formula, which is obtained through the standard Euclidean path integral method with imaginary time $\tau(\equiv it)$ [22]. The whole range of imaginary time $0 \le \tau \le \beta$ ($\beta \equiv 1/k_BT$) is divided into M slices of equal size $\Delta\tau = \beta/M$. Then the completeness relation $\prod_i \sum_{S_i=\pm 1} |S_i\rangle\langle S_i| = 1$ is inserted between neighboring slices, where $|S_i\rangle$ is an eigenstate of σ_i^z with the ith spin pointing up ($S_i = +1$) or down ($S_i = -1$). In order to choose the right number of slices M, we have computed physical quantities with different values of M for each given parameter set. It turns out that there is a lower limit M_c above which the result becomes independent of M. This indicates that ξ_τ, the correlation length in the imaginary time direction, is a finite fraction of β. Typically, $M_c \sim 30$, but it becomes larger for very low temperatures. More details on the method used here may be found in reference [18]. We have employed the cluster algorithm [23] for efficiency. All results were obtained after averaging over a large number of randomly generated static SF networks until a desired statistical precision was achieved.

3 Quantum fluctuations in the whole network

We first analyze the case where a uniform transverse magnetic field is applied to the whole network, i.e.,

$$\Delta_i = \Delta \text{ for all } i. \tag{4}$$

For simplicity, we have also restricted the analysis to $\gamma = 6$. It is known that for $\gamma > 5$, the ferromagnetic

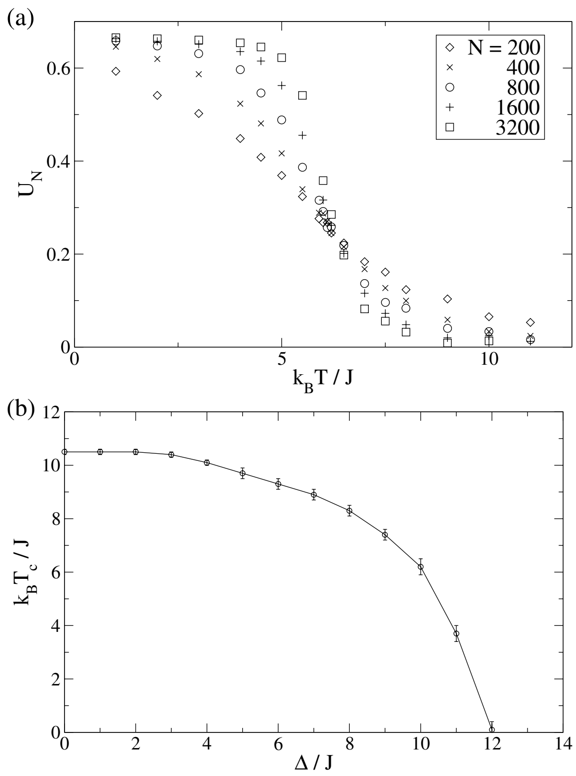

Fig. 1. (a) The fourth order Binder cumulant U_N is drawn as a function of temperature for five different system sizes N. The parameters used are $\gamma = 6$, $\Delta/J = 10$, and $M = 50$. Error bars are omitted, but they are about the same size as the symbols. For large N, the curves cross at a single point, which is identified as the ferromagnetic-paramagnetic phase transition temperature T_c. In this particular data set, $k_B T_c/J = 6.2 \pm 0.2$. (b) Transition temperature relative to the Ising coupling strength $k_B T_c/J$ is drawn as a function of transverse field strength Δ for SF networks with $\gamma = 6$. From extrapolation, it appears to vanish at $\Delta_c/J \gtrsim 12$. The system is in the ferromagnetic(paramagnetic) phase below (above) the curve.

Ising model in the absence of the quantum fluctuations will exhibit critical behaviour which belongs to the mean-field universality class [12–14]. In fact, our model reverts to the classical one if $\Delta = 0$ and this reversion has been used to test our Monte Carlo simulation method. All results of the zero-field classical case agree well with the predictions of the earlier studies.

We have determined the transition temperature as a function of transverse field, $T_c(\Delta)$, from the finite-size scaling method. For a given Δ, we have varied the temperature T and computed the fourth order Binder cumulant [24] of the magnetization m

$$U_N(T) = 1 - \frac{[\langle m^4 \rangle]}{3[\langle m^2 \rangle]^2} \tag{5}$$

for several systems with different numbers of spins N. Here, two different brackets $\langle \cdots \rangle$ and $[\cdots]$ are used to denote the thermal and the network configuration average, respectively. Then, T_c is determined from the crossing point of the graphs $U_N(T)$. An example of such a graph at $\Delta/J = 10$ is shown in Figure 1a.

By repeating the above analysis with varying Δ/J, one can determine T_c as a function of Δ/J. The result is plotted in Figure 1b. The figure can also be interpreted as a phase diagram, since the system is in the ferromagnetic (paramagnetic) phase below (above) the curve in the thermodynamic limit. Note that the transition temperature T_c decreases as Δ increases, which implies that quantum fluctuations weaken ferromagnetic order. It appears that T_c eventually vanishes at a quantum critical point $\Delta_c/J \gtrsim 12$ [25]. In our Monte Carlo simulation method, however, the exact value of this critical field Δ_c may only be estimated from the extrapolation of the $T_c(\Delta)$ curve, since the method fails at exactly zero temperature ($\beta \to \infty$) where an infinite number of imaginary time slices would be required. In our analysis, we were able to approach this quantum critical point down to $k_B T_c/J = 0.1 \pm 0.3$ at $\Delta/J = 12$, but no further, due to the above mentioned limitations of the method used. The quantum critical point is in itself an intriguing subject, but it will not be discussed further in this paper.

We now turn to the calculation of the critical exponents which identify the universality class of the phase transitions. By comparing the mean-field critical exponents of the purely classical case with those of the transitions at finite transverse fields, we can tell whether quantum fluctuations alter the universality class or not. First, the critical exponent $\bar{\nu}$, which describes the divergence of the correlation volume at T_c [26], can be extracted from the finite-size scaling formula of the Binder cumulant

$$U_N(T, N) = \tilde{U}((T - T_c)N^{1/\bar{\nu}}). \tag{6}$$

Near T_c, the above expression may be expanded as

$$U_N(T, N) = U^* + U_1 \left(1 - \frac{T}{T_c}\right) N^{1/\bar{\nu}}, \tag{7}$$

where U^* and U_1 are constants. We thus get the relation

$$\Delta U_N \equiv U_N(T_1, N) - U_N(T_2, N) \propto N^{1/\bar{\nu}}, \tag{8}$$

where T_1 and T_2 are close to T_c. This relation can be used to obtain $\bar{\nu}$ from the data. The other critical exponents are then obtained from the specific heat c, magnetization m, and susceptibility χ using the following finite-size scaling formulas:

$$c(T, N) = N^{\alpha/\bar{\nu}} \tilde{c}((T - T_c)N^{1/\bar{\nu}}) \tag{9}$$
$$m(T, N) = N^{-\beta/\bar{\nu}} \tilde{m}((T - T_c)N^{1/\bar{\nu}}) \tag{10}$$
$$\chi(T, N) = N^{\bar{\gamma}/\bar{\nu}} \tilde{\chi}((T - T_c)N^{1/\bar{\nu}}). \tag{11}$$

An example of the result from the finite-size scaling analysis at $\Delta/J = 10$ is shown in Figure 2. For each physical quantity, the curves which represent different system sizes clearly collapse to one single curve near T_c. For all values of Δ/J that appear in Figure 1b, we have confirmed that the critical exponents are: $\bar{\nu} = 2.0 \pm 0.1$, $\alpha = 0.0 \pm 0.1$, $\beta = 0.5 \pm 0.1$, and $\bar{\gamma} = 1.0 \pm 0.1$. Within the error bars,

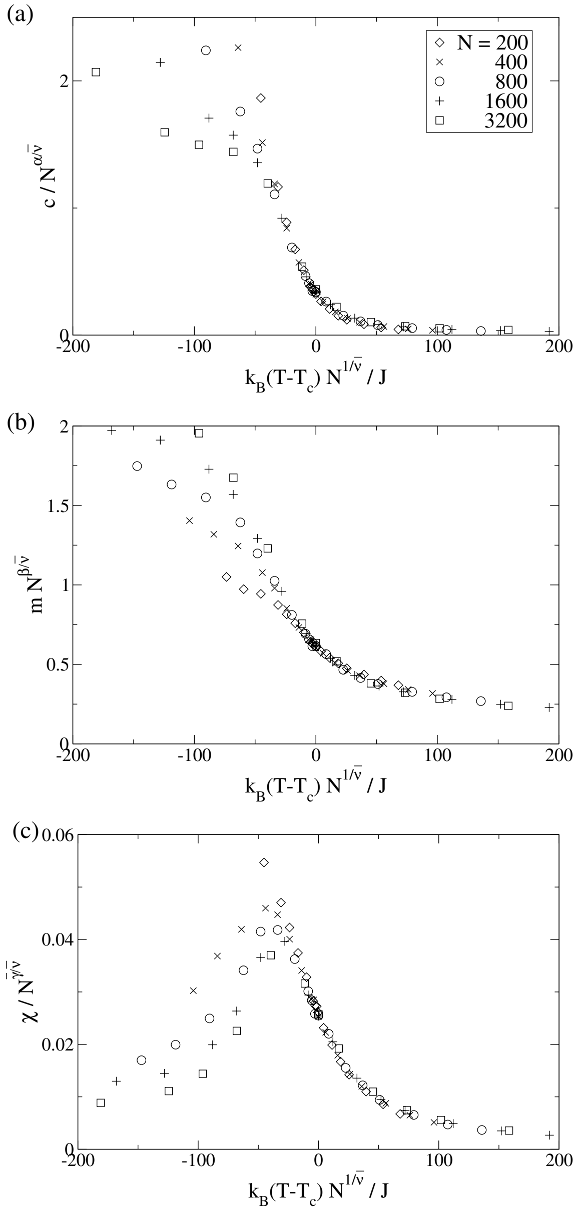

Fig. 2. Universal scaling functions for (a) specific heat, (b) magnetization, and (c) susceptibility. These data sets are for $\gamma = 6$, $\Delta/J = 10$, $k_B T_c/J = 6.2$, and $M = 50$. We used the mean-field critical exponents $\bar{\nu} = 2$, $\alpha = 0$, $\beta = 1/2$, and $\bar{\gamma} = 1$ here. The legend in (a) is common to all three figures. Near $T = T_c$, all data appear to collapse to one single curve for each physical quantity.

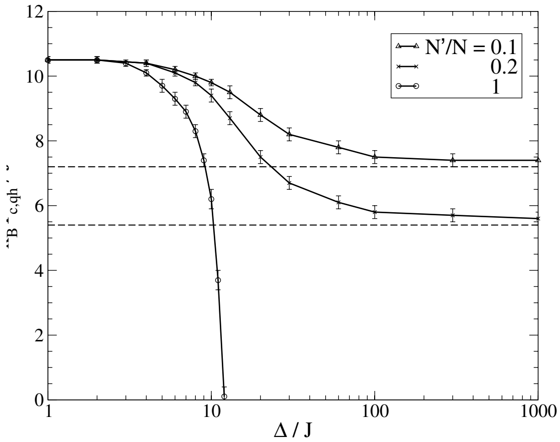

Fig. 3. Curves of the ferromagnetic-paramagnetic phase transition temperature $T_{c,\mathrm{qh}}(\Delta)$ of the "quantum-hub model". The transverse magnetic field Δ is applied exclusively at N' hub nodes. The horizontal axis is drawn in logscale. The result of the original uniform-field model ($N'/N = 1$) is also drawn for comparison. At large Δ, the results for $N'/N = 0.1$ and 0.2 appear to saturate at 7.2 and 5.4, respectively, which are the results for the classical "removed-hub model" and are drawn in the figure as horizontal dashed lines.

they are the same as the mean-field critical exponents: $\bar{\nu} = 2$, $\alpha = 0$, $\beta = 1/2$, and $\bar{\gamma} = 1$. Therefore, we may conclude that the quantum fluctuations induced by a uniform transverse magnetic field does not change the universality class of the ferromagnetic-paramagnetic phase transition.

The above results are very similar to those obtained in the case of small-world networks [18]. This may be because the two problems belong to the same mean-field universality class when $\gamma > 5$. However, the two models are also very different in many ways even for $\gamma > 5$. One such difference is that there are hubs in scale-free networks that play very important roles in mediating the exchange of ordering information between nodes. The roles of the hubs are the subject of the following section.

4 Quantum fluctuations only at hub nodes

In this section, we will investigate whether and how the influence of hub spins to the whole system is weakened by quantum fluctuations. The transverse magnetic field is now turned on only at a small number N' of nodes that have more links than the others.

$$\Delta_i = \begin{cases} \Delta, & i \text{ is a hub node,} \\ 0, & \text{otherwise.} \end{cases} \qquad (12)$$

We have performed the Monte Carlo simulation with various different fractions N'/N. Sample results for $N'/N = 0.1$ and 0.2 are shown in Figure 3. At $\Delta = 0$, the model is again purely classical and the curves for different N' all converge to the classical result. As Δ grows, the transition temperature $T_{c,\mathrm{qh}}(\Delta)$ decreases, but saturates at a finite value $T_{c,\mathrm{qh}}(\infty)$ at large Δ. (The letters 'qh' in the subscript is short-hand notation for 'quantum hubs'.) As the number of hubs N' increases, the saturation temperature $T_{c,\mathrm{qh}}(\infty)$ decreases. Eventually at $N' = N$, the model becomes identical to the model discussed in the previous section where the transverse field is applied to the entire system.

For the purpose of comparison, we have also performed a similar analysis by turning on the transverse field only at the N' nodes that have the *fewest* links. For the same fractions $N'/N = 0.1$ and 0.2, it was not possible to detect any change in T_c even at a transverse field as large as

$1000J$. Therefore it is evident that the above behaviour of decreasing $T_{c,\mathrm{qh}}(\Delta)$ is due to the fact that the quantum fluctuations are applied to the spins at the hubs. We may thus conclude that the hub spins play an important role in maintaining ferromagnetic order in the whole system, and that $T_{c,\mathrm{qh}}(\Delta)$ decreases with Δ because the quantum fluctuations lessen the effect of hub spins.

The saturation of $T_{c,\mathrm{qh}}(\Delta)$ at large Δ may be easily understood if we compare the current model with another one: a completely classical system ($\Delta = 0$) with the hub nodes simply *removed* from the network. It turns out that the transition temperature $T_{c,\mathrm{rh}}$ for the new model (subscript 'rh' for 'removed hubs') is the same as the saturation value $T_{c,\mathrm{qh}}(\infty)$ of the above quantum hub model. The reason for this is quite simple: at strong transverse field Δ, quantum fluctuations completely wipe out the average z-direction component of the hub spins so that they can neither affect other spins nor contribute to magnetization.

An analysis of critical behaviour similar to the one in the previous section again shows that selectively turning on quantum fluctuations for hub spins does not alter the universality class of the phase transition.

Before concluding, we would like to make a few brief comments about the influence of quantum fluctuations for networks with $\gamma < 5$. It is well-known that as γ decreases the role of hubs become more important, and that the critical exponents take different values for $\gamma < 5$, which means that the problem belongs to a different universality class than that of the simple mean-field. However, simply applying a uniform magnetic field to the whole system would probably not alter the universality class of the problem even for the case $\gamma < 5$. Of course, this hypothesis must actually be tested before making any conclusive statements, because so far only mean-field cases have been studied in connection to quantum fluctuations. It will be also interesting to see how a local magnetic field on the hubs will affect the system when $\gamma < 5$ and the role of hubs is enhanced.

5 Conclusions

We have performed a quantum Monte Carlo simulation analysis of a ferromagnetic Ising spin system connected through a SF network. We have considered a model in which one can control quantum fluctuations of each individual spin by adjusting a transverse magnetic field at each node. When there is a uniform transverse field Δ over the whole system, the ferromagnetic-paramagnetic phase transition occurs at temperature $T_c(\Delta)$ which decreases with growing Δ and finally vanishes at a finite Δ_c. The computed critical exponents for $\gamma > 5$ indicate that the model remains in the simple mean-field universality class even in the presence of a transverse field. We have also performed a similar analysis by turning on the transverse field at only a small number of hubs, i.e., nodes that have more links than the others. The transition temperature $T_{c,\mathrm{qh}}(\Delta)$ decreases with growing Δ, but saturates at a finite temperature $T_{c,\mathrm{qh}}(\infty)$. This saturation transition temperature has been found to be the same as the transition temperature of another model in which the hub nodes are simply removed from the network. In contrast, if we turn on the transverse field at the same number of nodes but at those which have the fewest links, the transition temperature is not affected by the quantum fluctuations. We have attributed this result to the special role played by the hub spins in maintaining ferromagnetic order in the whole system. Again, applying the transverse field selectively on the hub spins does not change the universality class of the phase transition.

The author benefited from valuable discussions with M.-S. Choi and H. Jeong. This work was supported by the Soongsil University Research Fund.

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